MATH1520 University Mathematics for Applications **Spring 2021**

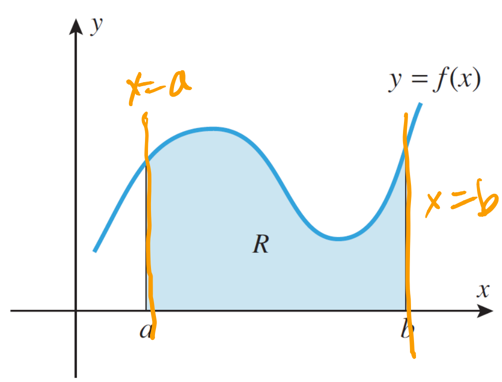
Chapter 10: Definite Integrals

Learning Objectives:
(1) Define the definite integral and explore its properties.
(2) State the fundamental theorem of calculus, and use it to compute definite integrals.
(3) Use integration by parts and by substitution to find integrals.
(4) Evaluate improper integrals with infinite limits of integration.

1 Riemann Sums & Definite Integrals

Suppose f is a function on $[a,b]$. Suppose further that $f(x)$ is positive on $[a,b]$. The we define

$$\int_a^b f(x)\,dx = \text{ area between } f(x) \text{ and the } x\text{-axis.}$$



What if some of the value of $f(x)$ is negative? Because $f(x)$ is negative, the "height" of $f(x)$ at this point is negative, so we take the area as negative. Therefore, we have the following definition.

Definition 1.1 (Total Signed Area). Let $y = f(x)$ be defined on a closed interval $[a,b]$. The **total signed area from** $x = a$ **to** $x = b$ **under** f is:

(area under f and above the x-axis on $[a,b]$) $-$ (area above f and under the x-axis on $[a,b]$).

the graph of. the graph

Geometric interpretation of integration The **definite integral of f on $[a, b]$** is the total signed area under f on from a to b, denoted

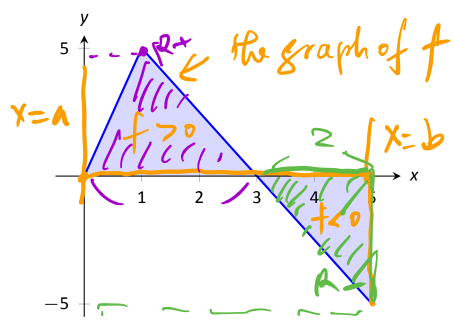
also called the "lower limit of the integral"
"upper limit"

$$\int_a^b f(x)\,dx,$$

where a and b are the **bounds (or limits) of integration**.

We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve".

Example 1.1. Consider the function f given below. Compute $\displaystyle\int_0^5 f(x)\,dx$.



The total signed area between the graph of f and the x-axis over the interval [a, b]
= Area of R+ − Area of R−
=

Solution. The graph of f is above the x-axis on $[0, 3]$. The area is $\frac{1}{2} \times 3 \times 5 = 7.5$. = Area(R+)

The graph of f is under the x-axis on $[3, 5]$. This is the "negative" area. The area is $-\frac{1}{2} \times 2 \times 5 = -5$. Hence

Area of R−

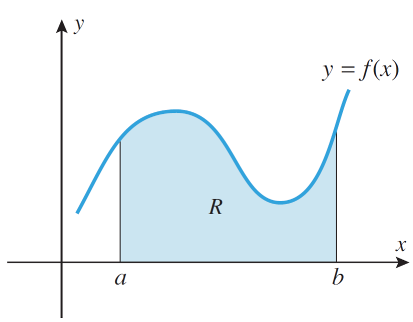
$$\int_0^5 f(x)dx = 7.5 - 5 = 2.5.$$

Area of R+

Area(R−) = ½(2)(5)
= 5

■

What ~~if the area is not regular, as the one shown~~ below?



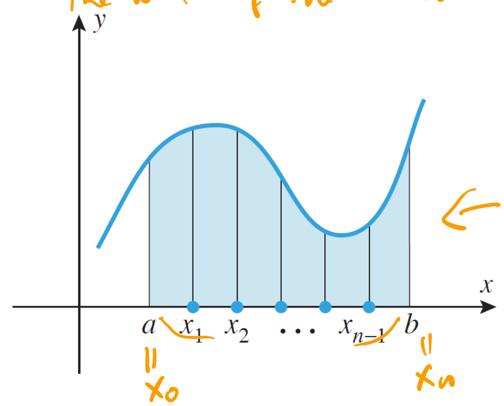
Idea: Approximate the area by small rectangles!

1. A partition of $[a, b]$: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$ divides $[a, b]$ into n subintervals $[a_{k-1}, a_k]$ with width:

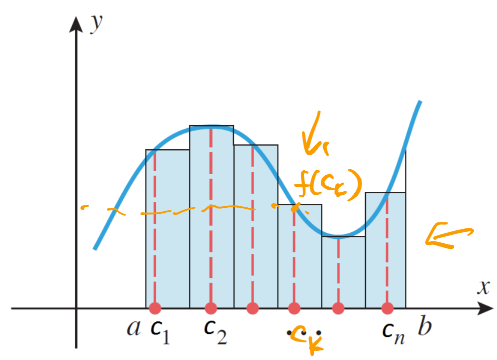
$x_0 = a, \ x_n = b$

$$\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}, \quad k = 1, 2, \ldots, n.$$

the width of the k-th interval $[x_{k-1}, x_k]$



2. Choose points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, to form small rectangles.

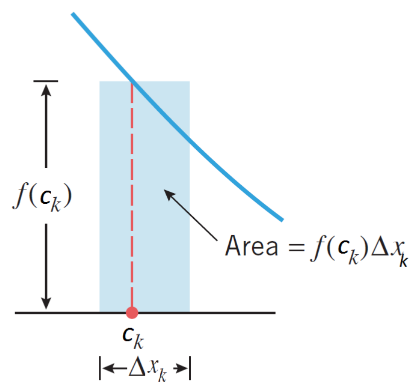


3. Calculate the area of each rectangle and sum them up.

   For the kth subinterval,

$$\boxed{\text{Area of } k\text{th rectangle} = \text{height} \times \text{width} = f(c_k)\Delta x_k}$$

$$f(c_k) \times \underbrace{(x_k - x_{k-1})}_{\Delta x_k}$$



Definition 1.2.

$$\boxed{\sum_{k=1}^{n} f(c_k)\Delta x_k} \approx \int_a^b f(x)\,dx$$

approximate

is called a Riemann Sum of f on $[a, b]$.

In particular,

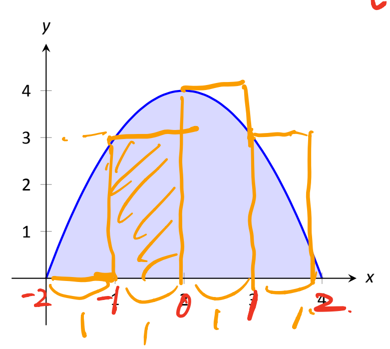
if	$c_k = x_{k-1}$, the sum is called	left Riemann sum
if	$c_k = x_k$,	right Riemann sum
if	$c_k = \dfrac{x_{k-1} + x_k}{2}$,	mid-point Riemann sum

Expectation: " "

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx$$

(as a signed area)

Example 1.2. Approximate the area under $y = 4 - x^2$ ($f(x)=$) on ~~$[0, 4]$~~ $[-2,2]$ with partition $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$.

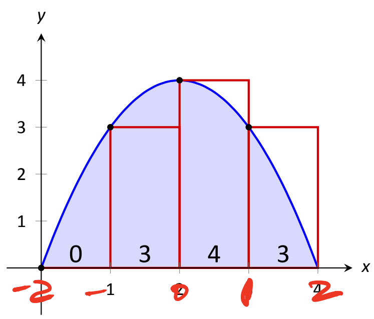


1. Left Riemann sum: $c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 3$.

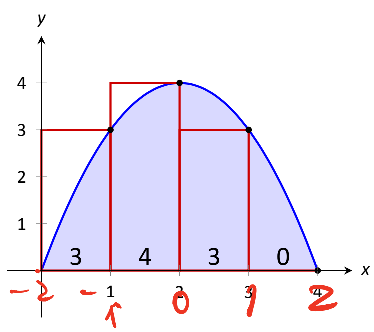
when $n=4$

$$\text{Area} \approx f(\cancel{0}\,{-2}) \cdot 1 + f(\cancel{1}\,{-1}) \cdot 1 + f(\cancel{2}\,0) \cdot 1 + f(\cancel{3}\,1) \cdot 1 = 10.$$

$f(-2) = 0$, $f(-1) = 3$, $f(0) = 4$, $f(1) = 3$



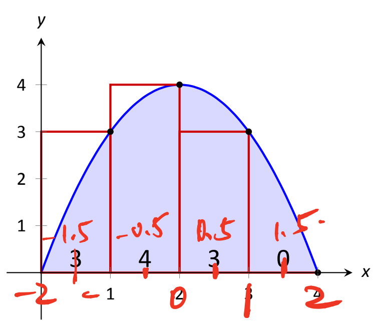
2. Right Riemann sum: $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4.$



$$\text{Area} \approx f(\cancel{1}\,{-1})\cdot 1 + f(\cancel{2}\,0)\cdot 1 + f(\cancel{3}\,1)\cdot 1 + f(\cancel{4}\,2)\cdot 1 = 10.$$

$= 3 \quad = 4 \quad = 3 \quad = 0$

3. Mid-point Riemann sum: $c_1 = , c_2 = 2, c_3 = 3, c_4 = 4.$



$$\text{Area} \approx f(\cancel{0.5}\,{-1.5})\cdot 1 + f(\cancel{1.5}\,{-0.5})\cdot 1 + f(\cancel{2.5}\,0.5)\cdot 1 + f(\cancel{3.5}\,1.5)\cdot 1 = 11.$$

Question: How to get better approximation of the area?

Solution: Increase number of rectangles.

width of each subinterval

Δx_k

Definition 1.3. Let $f(x)$ be continuous on $[a,b]$. Consider the partition: $x_k = \frac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$. For any $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, $\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$ is a fixed number, called definite (Riemann) integral of $f(x)$ on $[a,b]$, denoted by $\int_a^b f(x)\,dx$, i.e.,

"deforming" the summation notation as $n\to\infty$

$$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx}$$

"infinitesimal version of "Δ"

Hard Theorem: Let f be a piecewise continuous function, then $\int_a^b f(x)\,dx$ is well-defined. I.e. The limit in the preceding definition exists, and is independent of the choices of c_k.

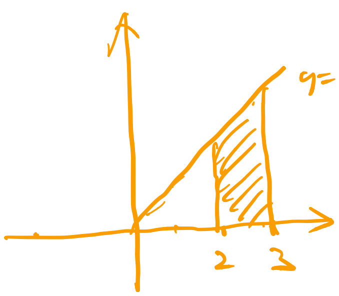
Remark. The "Lebesque integral" is well-defined for more general functions.

Example 1.3. Evaluate $\int_2^3 x\,dx$ using the left Riemann sum with n equally spaced subintervals.

n-th. left Riemann sum $\quad x_0 = 2, \quad x_n = 3$

$\Delta x_k = \frac{3-2}{n} = \frac{1}{n}.$

$x_k = 2 + k\cdot\Delta x_k = 2 + \frac{k}{n}.$



$$\sum_{k=1}^{n} f(x_{k-1})\Delta x_k$$
$$= \sum_{k=1}^{n} x_{k-1}\frac{1}{n}$$
$$= \frac{1}{n}\left(\sum_{k=1}^{n} 2 + \sum_{k=1}^{n} \frac{(k-1)}{n}\right)$$
$$= \frac{1}{n}\left(2n + \underbrace{\frac{0}{n} + \frac{1}{n} + \cdots \frac{n-1}{n}}\right)$$
$$= 2 + \frac{1}{n^2}\left(1+2+\cdots(n-1)\right)$$
$$= 2 + \frac{1}{n^2}\cdot\left(\frac{1+n-1}{2}\right)(n-1)$$
$$= 2 + \frac{1}{2}\frac{n-1}{n}$$

$$\int_2^3 f(x)\,dx = \int_2^3 x\,dx = \lim_{n\to\infty}\left(2 + \frac{1}{2}\frac{n-1}{n}\right) = 2 + \frac{1}{2} = \frac{5}{2}$$

$\square$

$f(x) = x^2$

Example 1.4. Evaluate $\int_0^1 x^2\,dx$ using the right Riemann sum with n equally spaced subintervals.

$x_0 = 0$ $\quad x_n = 1$

Solution. Consider the partition of $[0,1]$: $x_k = \frac{k}{n}$, $k = 0, \dots, n$.
Right Riemann sum: on $[x_{k-1}, x_k]$, $c_k = x_k = \dfrac{k}{n}$.

$$\sum_{k=1}^{n} f(\underset{x_k}{c_k})\Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \underbrace{\frac{1}{n}}_{\Delta x} = \frac{(n+1)(2n+1)}{6n^2}.$$

$$\left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

So, $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to+\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$ ■

Remark. It's so complicated to used definition to compute $\int_a^b f(x)\,dx$. Later, we will discuss another easier method: fundamental theorem of calculus.

Theorem 1.1 (Properties of definite integrals).

1. $\boxed{\int_a^a f(x)\,dx = 0}$
2. $\boxed{\int_a^b k\,dx = k(b-a)}$
3. $\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$
   $\boxed{kf(x)\,dx = k\int_a^b f(x)\,dx}$
4. *if* $a < b$,
   $\boxed{\int_b^a f(x)\,dx \triangleq -\int_a^b f(x)\,dx}$ ($\triangleq$, *defined to be*)
5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$
6. *if* $f(x) \le g(x)$ *on* $[a,b]$, *then*
   $\boxed{\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx}$